Random Matrices

→ Required student problem/topic presentation every class.

→ Either P/F. Require 2P/3 (or 66.6% or more).

- 3 grades A, A^-, B.

- I am not an expert. Teaching it because I'm writing a paper on RM.

- Requirements: 471,

Take $M \in \mathbb{C}^{n^2}$ st $M = M^* = \overline{(M^T)}$. This is a Hermitian matrix.

There are 3 big ensembles:

1) Orthogonal $M = \overline{M} = M^T$ (Real)

2) Hermitian

3) Symplectic: M is $2N \times 2N$. $M = M^*$ (so Hermitian)

But also $M = J M^T J^T$

↳ won't define. But related to Hermitian matrices that are quaternion real, & self dual.

A matrix ensemble is a measure on $\mathbb{R}^{n^2}$, (subject to various restrictions).

Typically, $P(M) \ll \text{Leb}(\mathbb{R})^{\otimes n^2}$ and so is defined via a density:

$$P(M) \prod_{i=1}^{N} \prod_{j=1}^{N} dM_{ij} = e^{-\text{tr}(V_{N,\beta}(M))} \prod_{ij} d_\beta M \quad \text{(*1)}$$

$V_{N,\beta}(x) = x^2$ Gaussian ensembles.

Note independence of entries.

→ In the Orthogonal case $d_\beta M$ must assign measure 1 to $\text{Im}(M_{ij}) = 0$, so we have written d_β like that.

☆ HW

→ For various physical reasons (see Ch2 of Mehta) we want ensembles of random matrices of $\circledast 1$ to be invariant under conjugation by

1) orthogonal matrices

2) Unitary matrices

3) Symplectic matrices.

<u>Lemma</u>: If $A = A^*$ then all eigenvalues are real:

$$Av = \lambda v \quad , \quad v^T A^T = \lambda v^T$$

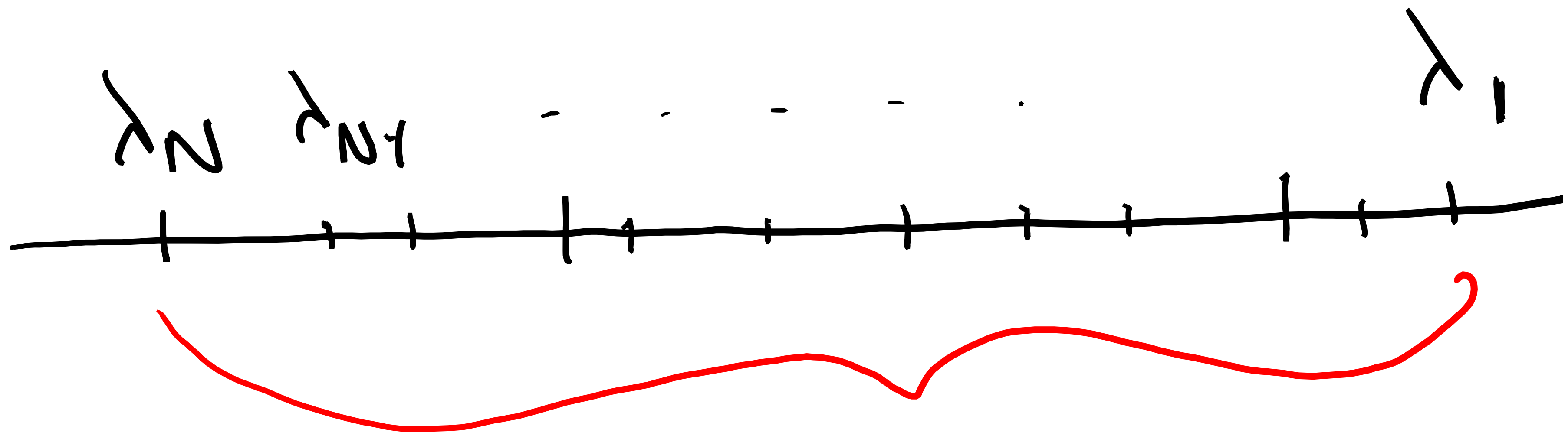
$$\overline{v^T}\,\overline{A^T} = \overline{\lambda}\,\overline{v^T} \iff v^* A^* = \overline{\lambda} v^*$$

$$\iff v^* A = \overline{\lambda} v^*$$

$$(v^* v)\lambda = v^* A v = \overline{\lambda}(v^* v)$$

$$\Rightarrow \quad \lambda = \overline{\lambda}$$

(Check yourself)



Random Realization

Only statistical properties are interesting.

For the Gaussian ensembles, we have

$$p_N(\lambda)\, d\lambda = \frac{1}{Z} \prod e^{-c_\beta \sum_{i=1}^{n} \lambda_i^2} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \, d\lambda_1 \cdots d\lambda_N$$

$$\{ \lambda : \lambda_1 \leq \lambda_2 \cdots \leq \lambda_N \}$$

$$P_N(M)\, dM \xrightarrow{\text{Eigenvalue mapping}} p_N(\lambda)\, d\lambda$$

We say, a system is modelled by RMT if various statistical averages match those of $p_N(\lambda)d\lambda$.

The most typical averages are k-point correlation fns:

$$R_1(\lambda_1) = N \int p_N(\lambda_1, \dots \lambda_N)\, d\lambda_2 \dots d\lambda_N$$

invariant under permutations.

How is this useful? Well, $(\lambda = (\lambda_1, \dots \lambda_N))$

$$E[\#\{i : \lambda_i \in B\}] = \sum_{i=1}^{n} \int 1_{\{\lambda_i \in B\}}(\lambda)\, p_N(\lambda_1, \dots \lambda_N)\, d\lambda$$

Consider $\displaystyle\int 1_{\{\lambda_k \in B\}}(\lambda)\, p_N(\lambda_1, \dots \lambda_N)\, d\lambda$ — $\star 2$

Make the COV $\lambda' = \sigma_{1k} \lambda$ where σ_{1k} is the permutation matrix that exchanges λ_1 and λ_k. Then

$$1_{\{\lambda_k \in B\}}(\sigma_{1k}^{-1} \lambda') = 1_{\{\lambda_1 \in B\}}(\lambda')$$

$$(\star 2) = \int 1_{\{\lambda_1 \in B\}}(\lambda')\, p_N(\sigma_{1k}^{-1} \lambda')\, d\lambda'$$

BUT $p_N(\sigma_{1k}^{-1} \lambda') = p_N(\lambda')$

Then,

$$E[\#\{i : \lambda_i \in B\}] = \int_B R_1(\lambda)\, d\lambda$$

Ex:

$$E[\#\{(i,j) : (\lambda_i, \lambda_j) \in B] = \int_B R_2(\lambda_1, \lambda_2)\, d\lambda_1 d\lambda_2$$

Turns out that for GUE:

$$R_m(\lambda_1, \dots, \lambda_m) = \det\left(K_N(\lambda_i, \lambda_j)\right)_{1 \le i,j \le m}$$

(Determinental formula)

Amazing fact:

centering

$$\lim_{N \to \infty} K_N\left(a + \frac{x}{\sigma_N},\ a + \frac{y}{\sigma_N}\right) = K_\infty(x-y)$$

scaling

$$K_\infty(x-y) = \frac{\sin \pi(x-y)}{\pi(x-y)}$$

and thus

$$\frac{1}{\sigma_N^2} R_2\left(a+\frac{x}{\sigma_N}, a+\frac{y}{\sigma_N}\right)$$

$$\longrightarrow \det \begin{pmatrix} K_\infty(0) & K_\infty(x-y) \\ K_\infty(x-y) & K_\infty(0) \end{pmatrix}$$

$$= 1 - \left[\frac{\sin(\pi(x-y))}{\pi(x-y)}\right]^2$$

By previous

$$E\left[\#\left\{(i,j) : \frac{a}{\sigma_N} < \lambda_i - \lambda_j < \frac{b}{\sigma_N}, \ |\lambda_i|, |\lambda_j| \leq \frac{t_N}{\sigma_N}\right\}\right]$$

$$\longrightarrow \int_a^b 1 - \left(\frac{\sin \pi r}{\pi r}\right)^2 dr$$

(Details in Deift)

Similarly:

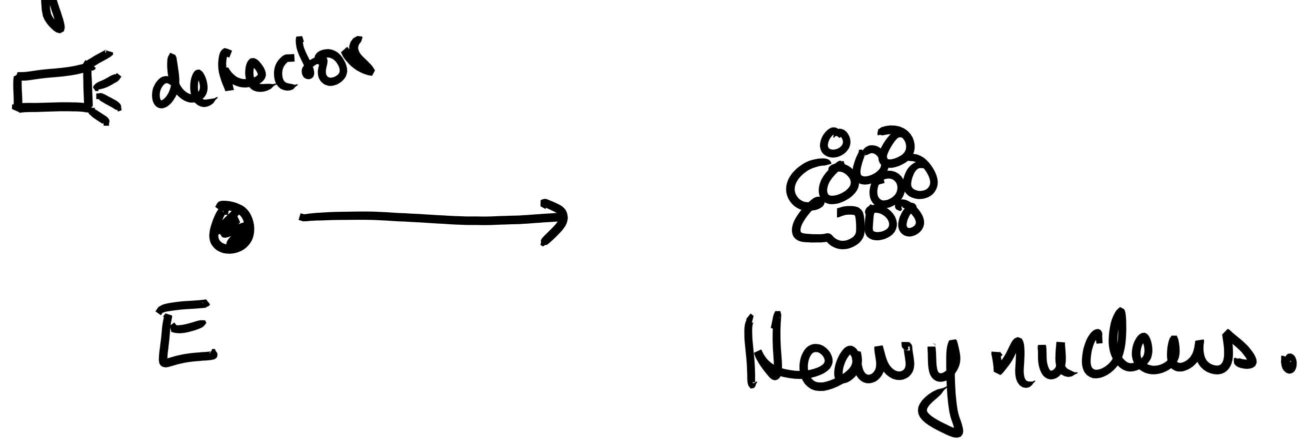
$$\lim_{N\to\infty} P_N\left(M: \frac{\lambda_1 - z_N}{\sigma_N} \le t\right) = \det\Big(1 - \underset{\substack{\uparrow \\ \text{Airy kernel}}}{\mathrm{Ai}}\Big)$$

$$= F_2(t) \quad \text{Tracy-Widom distribution.}$$

(called edge-statistics).

Examples:

1)



$E_1 \; \cdots \; E_N$ scattering resonances

Modulo symmetry considerations

→ All nuclei in the same class has same statistics as one of the 3 RMTs.

2) H. Montgomery

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad s \in \mathbb{C}$$

$$\zeta(s) = \text{Nice}(s) \quad \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\, \zeta(1-s)$$

zeros at all even integers

poles at all positive integers ≥ 1

Trivial zeros $s = -2n$

The positive zeros of the sin cancel with the poles of the gamma fn.

Nontrivial zeros occur on $\{ 1/2 + i\gamma_j \}$

(RH says all nontrivial zeros on critical line)

$$\tilde{\gamma}_j = \frac{\gamma_j \log \gamma_j}{2\pi} \qquad \text{(rescaled)}$$

$$\lim_{N \to \infty} \frac{\# \{ (j_1, j_2) : j_1 \neq j_2 \; 1 \leq j_1, j_2 \leq N, \; \tilde{\gamma}_{j_1} - \tilde{\gamma}_{j_2} \in (a,b) \}}{N}$$

$$= R(a,b)$$

$$\overset{?}{=} \int_a^b 1 - \left(\frac{\sin \pi r}{\pi r} \right)^2 dr$$

P3: Consider the following game.

$$\pi = 3 \quad 4 \quad 1 \quad 5 \quad 6 \quad 2$$

$$3 \qquad\qquad 3 \quad 4 \qquad\qquad \begin{array}{cc} 1 & 4 \\ 3 & \end{array} \qquad\qquad \begin{array}{ccc} 1 & 4 & 5 \\ 3 & & \end{array}$$

$$\begin{array}{cccc} 1 & 4 & 5 & 6 \\ 3 & & & \end{array} \qquad\qquad \underbrace{\begin{array}{cccc} 1 & 2 & 5 & 6 \\ 3 & 4 & & \end{array}}_{\#\text{ of piles } = \, q_N}$$

$$P\left(\frac{q_N - 2\sqrt{N}}{N^{1/6}} \leq t \right) = F_2(t)$$

Logan-Shepp-Vershik-Kerov. (Can be a whole course on rep theory by itself)

Bus Problem in Cuernavaca

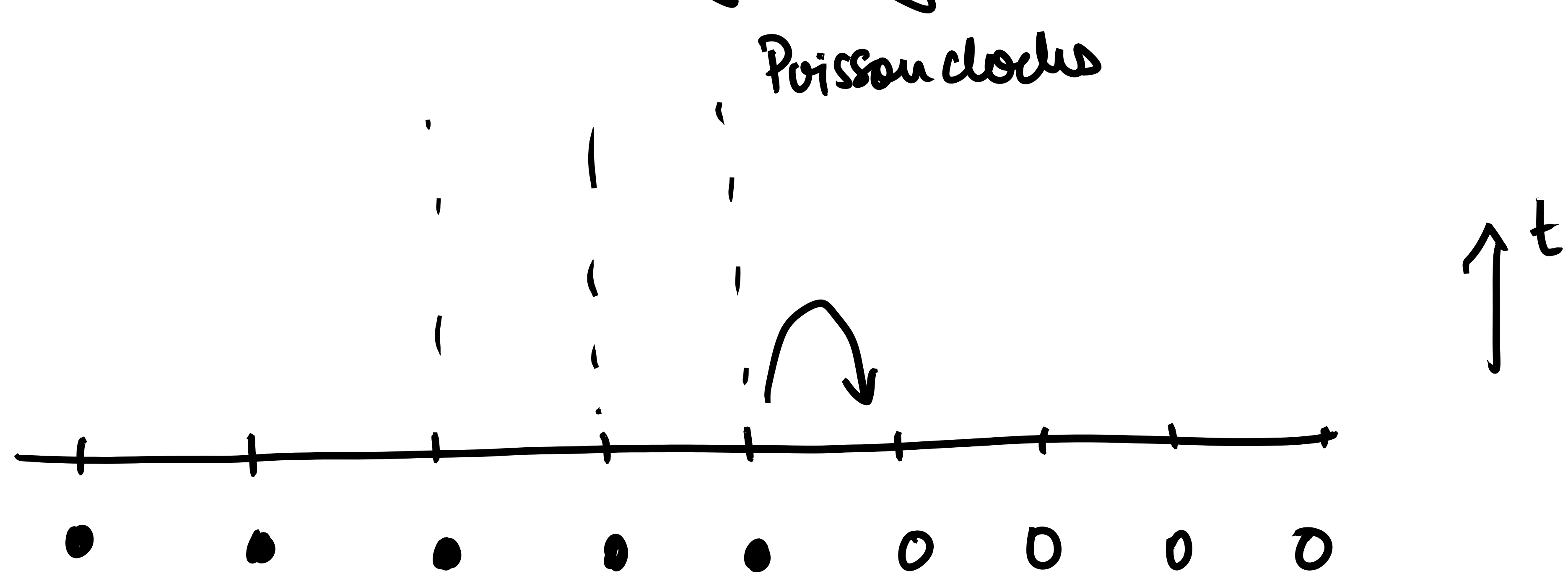
(City of population 1/2 a million)

Buses are privately run.

Too often buses bunch up. So they built recorders into each bus stop. So that the following driver could know when the bus up ahead passed the stop. Then the bus driver would adjust their speed.

"Repulsion is built in"

Vicious Walkers / Totally Asymmetric Simple Exclusion



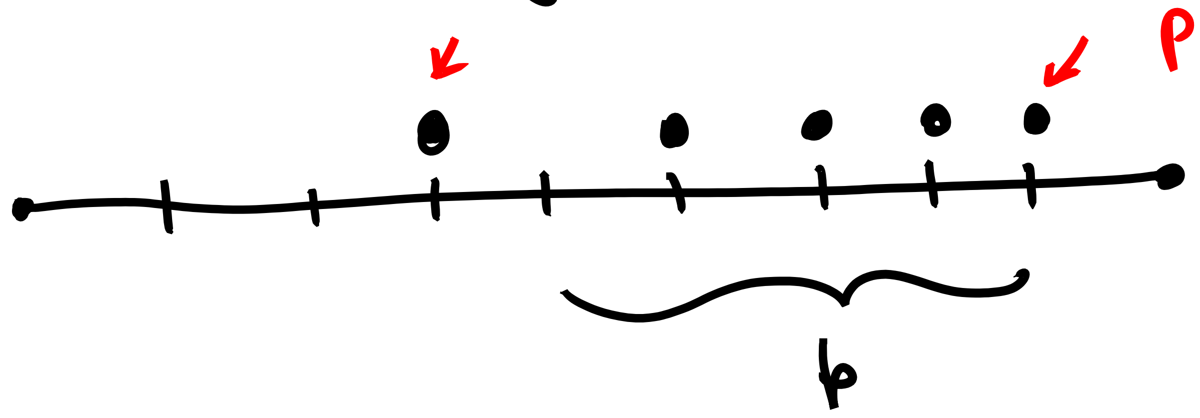
When clock rings, if no particle to the right particles jump.

Let d_N = location of leading particle at time N.

$$P\left(\frac{d_N - 2\sqrt{N}}{N^{1/6}} \leq x \right) = F_2(x)$$

Now, we can modify this model just a bit: fix p, # of particles

pick free particle with equal prob.



$$P\left(\frac{t_N - 2\sqrt{N}}{N^{1/6}} \leq x \right) = F_1(x)$$

$\beta = 1$

Exercises:

1) Lemma 2.6.3 in Mehta. All functions of an $N \times N$ matrix H that are invariant under non singular similarity transformations $H \longrightarrow A H A^{-1}$ can be expressed as a sum of traces of the first N powers of H.

2) Can you explain why these 3 ensembles of matrices arise in physics (Mehta Ch 2)? Why are these the only ensembles of interest there?

↳ Entries are indep

↳ Measure invariant under similarity trans of i) Unitary, Orthogonal, Symplectic